

CCS Frame: A Rigorous Foundation for Computable Coordinate Systems

Pan Guojun

March 24, 2026

Contents

Abstract	3
1. Introduction	4
1.1. The actual problem	4
1.2. Position of this paper	4
1.3. Contributions	4
1.4. In what sense CCS is paradigm-level	5
2. CCS as a Computable Object	5
2.1. Definition	5
2.2. Why scale belongs to the definition	5
2.3. Operational meaning	6
3. Surface-Adapted Scaled CCS	6
3.1. Adapted CCS	6
3.2. Tangent-scale frame and metric	6
4. Full CCS Variation and Its Exact Splitting	7
4.1. Full right-trivialized variation	7
4.2. Theorem 1: Exact CCS differential splitting	7
4.3. Consequence	7
5. From Full CCS Variation to the Induced Geometric Connection	8
5.1. What the full variation does not mean	8
5.2. Theorem 2: Natural projection from CCS to geometric connection	8
5.3. Interpretation of the exact curvature formula	8
6. Exact Curvature Formula at the Induced-Connection Level	9
6.1. Theorem 3: Exact curvature formula for the induced connection	9
6.2. Why unconditional pure-commutator curvature is false	9
7. Recovery of Complete Local Second-Order Geometry	9
7.1. Recovering the first fundamental form	9
7.2. Recovering the second fundamental form	9

7.3. Lemma: Basis-covariant recovery of the second fundamental form	10
7.4. Recovering the shape operator and curvatures	10
7.5. Gauge conventions	11
7.6. Proposition: Gauge transport between aligned and orthogonal adapted gauges	11
7.7. Theorem 4: Local second-order completeness of scaled adapted CCS	11
7.8. Definition: Local second-order CCS equivalence	12
7.9. Lemma: Constructive local realization of a second-order geometric class	12
7.10. Theorem 5: Local representation theorem for scaled adapted CCS	13
7.11. Corollary: CCS as a complete local second-order representation layer	13
7.12. What is and is not claimed	13
8. Discrete CCS and Second-Order Consistency	13
8.1. Discrete tangent-scale frame	13
8.2. Lemma: Discrete normal consistency	14
8.3. Lemma: Discrete second fundamental form consistency in a fixed gauge	14
8.4. Theorem 6: Discrete consistency of the CCS recovery chain	15
8.5. Consequence	15
9. Numerical Interpretation and Why CCS Can Be Stable	15
9.1. Implementation alignment	15
9.2. Equal-input equal-output principle	16
9.3. A local perturbation proposition	16
10. Why CCS Is More Than an Algorithmic Improvement	17
10.1. Classical pipeline versus CCS pipeline	17
10.2. Proposition 7: Object-level reformulation criterion	17
10.3. Corollary 8: CCS satisfies the object-level reformulation criterion	17
10.4. The real ceiling of the claim	18
11. Numerical Evidence	18
11.1. Parametric-surface benchmarks	18
11.2. Low-quality mesh robustness	18
11.3. Disciplined interpretation	19
12. Boundaries and Resolved Overclaims	19
12.1. Full right Maurer-Cartan form	19
12.2. Pure commutator curvature	19
12.3. General mesh / point-cloud theory	19
12.4. Blanket complexity declarations	19
13. Main Conclusion	19
14. Next Steps	20
Appendix A. Core theorem list	20

Abstract

This paper develops a rigorous geometric foundation for the **Computable Coordinate System** (CCS) and its induced **CCS frame** viewpoint. The central claim is that CCS should not be understood merely as a new curvature formula or a convenient frame-field implementation. Its more fundamental role is to provide an explicit *object theory* for the differential comparison of spatial vectors when **position**, **orientation**, and **scale** vary simultaneously. In this formulation, a local coordinate system is itself promoted to a first-class computable object,

$$C = \{o, s, R\}, \quad E = RD, \quad D = \text{diag}(s_1, s_2, s_3),$$

where $o \in \mathbb{R}^3$ is the origin, $R \in SO(3)$ is the directional frame, and $s \in (\mathbb{R}_{>0})^3$ is the positive scale vector. This scale-bearing structure is essential: CCS is not a standard orthonormal frame field with light algebraic decoration.

On this basis, the paper establishes four theorem-level results. First, the full differential variation of a scaled adapted CCS splits uniquely into intrinsic tangential connection, extrinsic bending, and scale evolution. Second, the full right-trivialized variation is shown to be a pure-gauge object at the ambient group level, but its metric-adapted tangential projection induces a genuine surface connection

$$\Gamma_\mu = g^{-1}T^\top(\partial_\mu T),$$

from which the exact curvature formula

$$\Omega_{uv}^\Gamma = \partial_u \Gamma_v - \partial_v \Gamma_u + [\Gamma_u, \Gamma_v]$$

holds under commuting local coordinates. Third, a scaled adapted CCS determines the complete local second-order geometry of a regular surface patch: the first fundamental form, second fundamental form, shape operator, Gaussian curvature, signed mean curvature, and principal curvatures are all recoverable from CCS data. With explicit gauge conventions, CCS yields a complete local second-order representation layer for surface geometry. Fourth, when a discrete CCS approximates a smooth CCS to second order, the induced connection, second fundamental form, shape operator, and curvature invariants are second-order consistent.

The theoretical chain is connected to computation. The paper analyzes the numerical realization embodied by the `coordsystem` Python library and explains why the CCS route is structurally stable: it estimates local geometry through frame evolution and induced geometric coefficients rather than relying exclusively on raw second differences of positions. This does **not** justify blanket complexity or universal mesh/point-cloud claims. It does, however, support a sharper statement: within the benchmark settings already studied, CCS exhibits consistently better robustness on low-quality discrete meshes. The supplemental experiments show an average $9.13\times$ improvement in the 95th-percentile Gaussian-curvature error, an average $3.06\times$ improvement for mean curvature, and a Gaussian-curvature sign accuracy increase from about 73.8% to about 89.4%.

The paper also resolves several overstatements that often arise around CCS. The full right Maurer-Cartan form does **not** directly produce nonzero surface curvature; unconditional pure-commutator curvature is false in general; a general mesh/point-cloud theory has not yet been completed; and broad asymptotic complexity declarations cannot be justified without a precise computational model. The main conclusion is therefore stronger and cleaner: CCS is a rigorous coordinate-system object theory for spatial-vector differential problems, and curvature recovery is one natural consequence of that theory rather than its sole definition.

1. Introduction

1.1. The actual problem

The point of departure is not “how to invent another curvature formula.” The deeper problem is:

How can one compare and differentiate spatial vectors in a computable way when position, orientation, and scale all change simultaneously?

In most classical formulations, the coordinate system is treated as background representation, while differential geometry is expressed through tensors, Christoffel symbols, and curvature objects derived from derivatives of a parametrization. That mathematical apparatus is complete, but from the perspective of a computable object theory it leaves two gaps.

First, the coordinate system itself is usually not treated as a primary object of computation. Second, there is often a gap between the representation layer and the actual numerical implementation layer: positions are sampled, derivatives are approximated, normals are reconstructed, and only then are curvature quantities inferred. CCS addresses this by turning the local coordinate system itself into a structured object that can be composed, compared, differentiated, and discretized.

1.2. Position of this paper

This paper takes the following position.

1. CCS is first an object theory and only then a curvature theory.
2. The scale component is fundamental and cannot be removed without changing the identity of CCS.
3. The main theorem layer should be developed for **scaled, geometrically adapted, orthogonal-direction CCS**.
4. Fully non-orthogonal CCS is a meaningful extension, but it is not the right main theorem layer for the present foundational paper.
5. The physics-oriented narrative is intentionally excluded here; the present goal is to harden the geometric foundation first.

1.3. Contributions

The paper makes the following contributions.

1. It formalizes CCS as a scale-bearing coordinate-system object

$$C = \{o, s, R\} \quad \text{or equivalently} \quad C = (o, E), \quad E = RD \in GL^+(3).$$

2. It proves an exact differential splitting theorem for the local variation of adapted scaled CCS.
3. It identifies the correct meaning of the CCS curvature formula: the exact expression

$$\partial_u G_v - \partial_v G_u + [G_u, G_v]$$

is valid only when G_μ denotes the **induced geometric connection coefficients**, not the full right-trivialized group coefficients.

4. It proves that scaled adapted CCS determines the complete local second-order geometry of a regular surface patch.
5. It establishes a discrete second-order consistency theorem for the CCS recovery chain.
6. It gives a disciplined numerical interpretation of existing experiments and implementation results, including a clear statement of what has and has not been proved.

1.4. In what sense CCS is paradigm-level

The phrase “paradigm-level innovation” should not be used loosely. In this paper it has a precise meaning. CCS is paradigm-level only if it changes the formal object of local geometric computation rather than merely changing one estimator inside an already fixed framework.

Under that criterion, CCS is paradigm-level in the following restricted but substantial sense:

1. the primary state changes from a parametrization-plus-derived-tensors view to a coordinate-system object view,
2. the local state variables change from raw position derivatives alone to a coupled object carrying origin, orientation, and scale,
3. the differential law changes from “differentiate the surface first, then build geometry” to “differentiate the coordinate-system object, then project geometry”,
4. the discrete target changes from approximating only positional derivatives to approximating the coordinate-system object and its induced recovery chain.

This is a genuine reformulation of the problem class. What is **not** claimed is that CCS has already displaced every existing method in differential geometry or geometry processing.

2. CCS as a Computable Object

2.1. Definition

A three-dimensional computable coordinate system is defined by

$$C = \{o, s, R\},$$

where

- $o \in \mathbb{R}^3$ is the origin,
- $s = (s_1, s_2, s_3) \in (\mathbb{R}_{>0})^3$ is the positive scale vector,
- $R = [e_1 \mid e_2 \mid e_3] \in SO(3)$ is an oriented orthonormal frame.

Let

$$D = \text{diag}(s_1, s_2, s_3), \quad E = RD.$$

Then CCS can equivalently be written as

$$C = (o, E), \quad E \in GL^+(3).$$

Its homogeneous representation is

$$\widehat{C} = \begin{pmatrix} E & o \\ 0^\top & 1 \end{pmatrix}.$$

The key point is that the object contains position, orientation, and scale *in one algebraic package*. This is the structural difference from an ordinary frame field.

2.2. Why scale belongs to the definition

The scale vector is not a cosmetic addition. Once CCS is intended to compare spatial vectors under changing local units and changing local tangent magnitudes, scale is part of the geometric object itself. Removing scale collapses the theory back to a narrower orthonormal-frame theory and loses the direct encoding of metric information that CCS is designed to preserve.

2.3. Operational meaning

The original CCS formulation is operational rather than merely descriptive. At the object level, one expects at least the following semantics:

- vector-action by a coordinate object,
- relative transformation between coordinate objects,
- coordinate-object composition,
- differential comparison between nearby coordinate objects.

This operational viewpoint is why the local variation of the coordinate object becomes the starting point of geometry in CCS.

3. Surface-Adapted Scaled CCS

Let $U \subset \mathbb{R}^2$ be a parameter domain and let

$$X : U \rightarrow \mathbb{R}^3$$

be a smooth regular surface patch.

3.1. Adapted CCS

A **surface-adapted scaled CCS** on U is a smooth field

$$C(u, v) = (o(u, v), R(u, v), D(u, v))$$

such that

- $o(u, v) = X(u, v)$,
- $R = [e_1 \mid e_2 \mid e_3] \in SO(3)$,
- e_1, e_2 span the tangent plane of X ,
- $e_3 = e_1 \times e_2$ is the unit normal,
- $D = \text{diag}(s_1, s_2, s_3)$ with $s_i > 0$.

For the main theorem layer, we assume that e_1, e_2 are orthogonal tangent directions. They need not coincide with the raw coordinate directions (X_u, X_v) .

3.2. Tangent-scale frame and metric

Define the scaled tangent basis vectors

$$\tau_1 = s_1 e_1, \quad \tau_2 = s_2 e_2,$$

and the tangent-scale frame matrix

$$T = [\tau_1 \ \tau_2] \in \mathbb{R}^{3 \times 2}.$$

Then the induced metric on the tangent bundle is

$$g = T^\top T.$$

In the orthogonal-direction CCS gauge,

$$g = \begin{pmatrix} s_1^2 & 0 \\ 0 & s_2^2 \end{pmatrix}.$$

Thus metric data are already contained in the CCS object through the scale-bearing tangent basis.

4. Full CCS Variation and Its Exact Splitting

4.1. Full right-trivialized variation

Define the full right-trivialized variation by

$$G_\mu := (\partial_\mu \widehat{C}) \widehat{C}^{-1}, \quad \mu \in \{u, v\}.$$

Writing $\widehat{C}$ in block form gives

$$G_\mu = \begin{pmatrix} B_\mu & t_\mu \\ 0^\top & 0 \end{pmatrix},$$

with

$$B_\mu = (\partial_\mu E) E^{-1}, \quad t_\mu = \partial_\mu o - (\partial_\mu E) E^{-1} o.$$

4.2. Theorem 1: Exact CCS differential splitting

Theorem 1. Let $C = (o, R, D)$ be a smooth surface-adapted scaled CCS. Then

$$B_\mu = A_\mu + S_\mu,$$

where

$$A_\mu = (\partial_\mu R) R^\top \in \mathfrak{so}(3), \quad S_\mu = R((\partial_\mu D) D^{-1}) R^\top \in \text{Sym}(3).$$

Moreover, A_μ has the unique Darboux-type decomposition

$$A_\mu = \begin{pmatrix} 0 & -\alpha_\mu & -\beta_\mu \\ \alpha_\mu & 0 & -\gamma_\mu \\ \beta_\mu & \gamma_\mu & 0 \end{pmatrix}.$$

Hence the local CCS variation splits uniquely into:

1. intrinsic tangential connection coefficients α_μ ,
2. extrinsic bending coefficients β_μ, γ_μ ,
3. scale-evolution coefficients encoded in S_μ .

Proof. Since $E = RD$,

$$(\partial_\mu E) E^{-1} = (\partial_\mu R) D D^{-1} R^\top + R(\partial_\mu D) D^{-1} R^\top = (\partial_\mu R) R^\top + R(\partial_\mu D) D^{-1} R^\top.$$

Because $R \in SO(3)$, the matrix $(\partial_\mu R) R^\top$ is skew-symmetric. Because $(\partial_\mu D) D^{-1}$ is diagonal, its conjugate by R is symmetric. The decomposition of a matrix into skew-symmetric and symmetric parts is unique, which proves uniqueness. The Darboux form is the standard coordinate expression of a skew-symmetric 3×3 matrix. $\square$

4.3. Consequence

The theorem makes clear that scale evolution is a coequal differential layer, not an afterthought. Ordinary orthonormal frame theories directly carry the first two layers; CCS carries all three.

5. From Full CCS Variation to the Induced Geometric Connection

5.1. What the full variation does not mean

The object $G_\mu = (\partial_\mu \widehat{C})\widehat{C}^{-1}$ is the full right-trivialized variation of the coordinate-system object. At the ambient group level it satisfies a Maurer-Cartan flatness identity. Therefore:

the full right-trivialized CCS variation is **not** itself the nonzero surface curvature.

This point is essential. Scale does not rescue the incorrect statement that the full Maurer-Cartan form directly yields surface curvature.

5.2. Theorem 2: Natural projection from CCS to geometric connection

Theorem 2. Let $C = (o, R, D)$ be a surface-adapted scaled CCS and let

$$T = [\tau_1 \ \tau_2] = [s_1 e_1 \ s_2 e_2], \quad g = T^\top T.$$

On the region where τ_1, τ_2 are linearly independent, define

$$\Gamma_\mu := g^{-1} T^\top (\partial_\mu T) \in \mathfrak{gl}(2).$$

Then:

1. Γ_μ is the tangential connection induced by CCS,
2. Γ_μ depends only on the tangent-scale part of CCS and the metric compatibility encoded by g ,
3. Γ_μ is obtained by restricting the linear part of the full variation to the tangent-scale subbundle and expressing the tangential component in the basis $\{\tau_1, \tau_2\}$,
4. when $D = I$ and $T = [e_1 \ e_2]$, the formula reduces to the usual connection matrix of an adapted orthonormal tangent frame.

Proof. Since $\{\tau_1, \tau_2\}$ is a basis of the tangent plane, the tangential component of $\partial_\mu \tau_a$ has a unique expansion

$$\Pi_T(\partial_\mu \tau_a) = \Gamma_{\mu a}^b \tau_b.$$

Taking inner products with τ_c gives

$$\langle \tau_c, \partial_\mu \tau_a \rangle = g_{cb} \Gamma_{\mu a}^b.$$

Multiplying by g^{-1} yields

$$\Gamma_\mu = g^{-1} T^\top (\partial_\mu T).$$

Now $\partial_\mu T$ is exactly the derivative of the tangent-scale part of the CCS linear block $E = RD$. Projecting $\partial_\mu \tau_a$ to the tangent plane and re-expanding in the basis $\{\tau_1, \tau_2\}$ therefore gives the restriction of the full CCS variation to the tangent-scale subbundle. This coefficient matrix is exactly the induced tangential geometric connection. $\square$

5.3. Interpretation of the exact curvature formula

When one writes

$$\partial_u G_v - \partial_v G_u + [G_u, G_v],$$

the formula can represent genuine surface curvature **only if** G_μ denotes the induced geometric connection coefficients, i.e. $G_\mu = \Gamma_\mu$. It is false if G_μ is interpreted as the full right-trivialized group coefficients.

6. Exact Curvature Formula at the Induced-Connection Level

6.1. Theorem 3: Exact curvature formula for the induced connection

Theorem 3. Under the assumptions of Theorem 2, define the matrix-valued 1-form

$$\Gamma = \Gamma_u du + \Gamma_v dv.$$

If the local coordinate vector fields commute,

$$[\partial_u, \partial_v] = 0,$$

then the curvature 2-form of the induced connection is

$$\Omega^\Gamma = d\Gamma + \Gamma \wedge \Gamma,$$

and its (u, v) coefficient is

$$\Omega_{uv}^\Gamma = \partial_u \Gamma_v - \partial_v \Gamma_u + [\Gamma_u, \Gamma_v].$$

Proof. This is the standard curvature formula for a matrix-valued connection 1-form. Under commuting coordinates,

$$d\Gamma = (\partial_u \Gamma_v - \partial_v \Gamma_u) du \wedge dv,$$

and

$$\Gamma \wedge \Gamma = [\Gamma_u, \Gamma_v] du \wedge dv.$$

Adding the two terms yields the result. $\square$

6.2. Why unconditional pure-commutator curvature is false

The term $[\Gamma_u, \Gamma_v]$ alone does not in general represent the full curvature. It becomes the whole expression only under additional special conditions, for example if derivative terms vanish in a chosen gauge. Therefore the slogan “pure commutator curvature” is false as an unconditional theorem.

7. Recovery of Complete Local Second-Order Geometry

7.1. Recovering the first fundamental form

By construction,

$$g = T^\top T,$$

so the first fundamental form is immediately determined by the CCS tangent-scale frame.

7.2. Recovering the second fundamental form

Let $n = e_3$ be the unit normal. Since $\tau_a = s_a e_a$, the normal component of $\partial_\mu \tau_a$ satisfies

$$b_{\mu a} := \langle \partial_\mu \tau_a, n \rangle.$$

Because $\partial_\mu (s_a e_a) = (\partial_\mu s_a) e_a + s_a \partial_\mu e_a$ and $\langle e_a, n \rangle = 0$, one has

$$b_{\mu a} = s_a \langle \partial_\mu e_a, n \rangle.$$

Hence the normal-coupling terms of the rotational component A_μ determine the second-order bending coefficients.

If one chooses the tangent directions so that e_1, e_2 are aligned with the parameter directions (u, v) after normalization, then

$$L = \langle X_{uu}, n \rangle = b_{u1}, \quad M = \langle X_{uv}, n \rangle = b_{v1} = b_{u2}, \quad N = \langle X_{vv}, n \rangle = b_{v2}.$$

In a more general orthogonal-direction adapted gauge, let $P(u, v) \in GL(2)$ be the change-of-basis matrix from the raw coordinate tangent basis $\{X_u, X_v\}$ to the CCS tangent basis $\{\tau_1, \tau_2\}$. If $\widetilde{B}$ denotes the second fundamental form in the CCS tangent basis, then the coordinate-basis second fundamental form is

$$B = P^\top \widetilde{B} P.$$

Therefore once the tangent basis is fixed, or equivalently once P is known from the chosen gauge, the second fundamental form in any desired tangent basis is determined by CCS. In particular, the symmetric bilinear form

$$\widetilde{B} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

is determined in the aligned gauge, and transforms canonically under tangent-basis change.

7.3. Lemma: Basis-covariant recovery of the second fundamental form

Lemma 4. Let $\widetilde{B}$ be the second fundamental form represented in the CCS tangent basis $\{\tau_1, \tau_2\}$, and let $P \in GL(2)$ be the change-of-basis matrix from the coordinate tangent basis $\{X_u, X_v\}$ to $\{\tau_1, \tau_2\}$. Then:

1. $\widetilde{B}$ is uniquely determined by the normal components of the differentiated CCS tangent-scale vectors,
2. the coordinate-basis second fundamental form is

$$B = P^\top \widetilde{B} P,$$

3. therefore the second fundamental form recovered from CCS is basis-covariant and gauge-consistent once P is fixed.

Proof. The coefficients of $\widetilde{B}$ are exactly the values of the symmetric bilinear form II on the basis vectors of the CCS tangent basis, hence are uniquely determined once the normal components of $\partial_\mu \tau_a$ are known. If a tangent vector has coordinate-basis components ξ and CCS-basis components $\tilde{\xi}$, then $\tau \tilde{\xi} = X\xi$ and therefore $\tilde{\xi} = P\xi$. By bilinearity,

$$II(\xi, \eta) = \tilde{\xi}^\top \widetilde{B} \tilde{\eta} = \xi^\top P^\top \widetilde{B} P \eta,$$

which yields $B = P^\top \widetilde{B} P$. This proves basis covariance. $\square$

7.4. Recovering the shape operator and curvatures

Once g and B are known,

$$S = g^{-1} B$$

is the shape operator. The classical invariants follow:

$$K = \det(S), \quad H = \frac{1}{2} \text{tr}(S), \quad k_{1,2} = H \pm \sqrt{H^2 - K}.$$

7.5. Gauge conventions

To state a representation theorem precisely, one must mod out geometric redundancies. For the present paper, the relevant local redundancies are:

1. ambient rigid motions in $\mathbb{R}^3$,
2. local reparametrizations of (u, v) that preserve regularity,
3. tangent-basis normalization conventions,
4. the choice of normal orientation $n \mapsto -n$.

If these gauge freedoms are fixed by the conventions

- $o = X(u, v)$,
- $e_3 = e_1 \times e_2$,
- e_1, e_2 are orthogonal tangent directions with prescribed orientation,
- $s_a = \|\tau_a\|$,

then the remaining CCS data carry precisely the local geometric information claimed below.

7.6. Proposition: Gauge transport between aligned and orthogonal adapted gauges

Suppose a regular surface patch is described in the aligned gauge, so that the CCS tangent directions are the normalized parameter directions. Let $\{\tau_1, \tau_2\}$ be any other orthogonal adapted CCS tangent basis, and let $P \in GL(2)$ be the change-of-basis matrix from $\{X_u, X_v\}$ to $\{\tau_1, \tau_2\}$. Then:

1. the metric transforms by

$$\tilde{g} = P^\top g P,$$

2. the second fundamental form transforms by

$$\tilde{B} = P^\top B P,$$

3. the corresponding shape operators are conjugate,

$$\tilde{S} = P^{-1} S P,$$

4. therefore the invariant data (K, H, k_1, k_2) are unchanged.

This proposition makes explicit that aligned and orthogonal adapted gauges are different representatives of the same local second-order geometry class. The choice of gauge changes coordinates of the representation, not the geometric content.

7.7. Theorem 4: Local second-order completeness of scaled adapted CCS

Theorem 4. Let $C = (o, R, D)$ be a smooth surface-adapted scaled CCS on a regular surface patch, and assume that the tangent-scale frame is nondegenerate. Fix either:

1. the aligned gauge in which the CCS tangent directions are the normalized parameter directions, or
2. an orthogonal adapted gauge together with the tangent change-of-basis matrix P relating $\{X_u, X_v\}$ and $\{\tau_1, \tau_2\}$.

Then the CCS data determine uniquely the local second-order geometric data of the patch:

1. the first fundamental form $I = g$,
2. the second fundamental form $II = B$,
3. the shape operator $S = g^{-1}B$,
4. the Gaussian curvature K ,
5. the signed mean curvature H ,
6. the principal curvatures k_1, k_2 .

Moreover, after fixing the gauge conventions of Section 7.5, scaled adapted CCS provides a complete local second-order representation layer for surface geometry.

Proof. The metric g is recovered from $T^\top T$. In the aligned gauge, the coefficients of the second fundamental form are the normal components $b_{\mu a}$ computed above, so $\widetilde{B}$ is directly recovered. In a general orthogonal adapted gauge, the same normal-coupling data recover $\widetilde{B}$ in the CCS tangent basis, and the fixed matrix P yields the coordinate-basis form $B = P^\top \widetilde{B} P$. Then $S = g^{-1}B$ is uniquely determined in the chosen basis. Its determinant and trace give K and H , and its eigenvalues give k_1, k_2 . After gauge conventions remove the rigid-motion, local-coordinate, normalization, and normal-sign redundancies, no further local second-order geometric invariant remains unencoded. $\square$

7.8. Definition: Local second-order CCS equivalence

Two surface-adapted scaled CCS fields C and C' are called **locally second-order equivalent** at a point if, after applying the gauge identifications of Section 7.5, they induce the same first fundamental form and the same second fundamental form at that point. Equivalently, they induce the same shape operator, and therefore the same local second-order surface geometry.

This equivalence relation is the natural notion of “same geometric content” for the theorem chain developed here: it removes representational redundancy while preserving all local second-order invariants.

7.9. Lemma: Constructive local realization of a second-order geometric class

Fix either the aligned gauge or an orthogonal adapted gauge together with a tangent basis map P . Let a regular local second-order geometry class be given by a pair (g, B) with g positive definite and B symmetric in the chosen gauge. Then one can construct locally a scaled adapted CCS representative $C = (o, R, D)$ whose induced data recover the prescribed class.

Construction.

1. Choose a tangent basis in the selected gauge whose Gram matrix is the prescribed metric g .
2. Normalize this basis to obtain orthonormal tangent directions e_1, e_2 and set $e_3 = e_1 \times e_2$.
3. Set the positive scales s_1, s_2 so that $\tau_a = s_a e_a$ reproduces the chosen tangent basis.
4. Set $R = [e_1 | e_2 | e_3]$ and $D = \text{diag}(s_1, s_2, s_3)$ with arbitrary positive normal scale s_3 .
5. Choose the normal-coupling coefficients of the differentiated tangent-scale frame so that the recovered second fundamental form equals the prescribed B , or equivalently the prescribed $\widetilde{B}$ in the CCS tangent gauge.

Because the data are local and the patch is regular, these choices determine a local CCS representative realizing the given second-order geometry class.

This lemma is the constructive content hidden inside the existence part of the representation theorem: admissible second-order geometric data can be encoded by a local CCS representative in the main theorem setting.

7.10. Theorem 5: Local representation theorem for scaled adapted CCS

Theorem 5. Let $\mathcal{G}_2$ denote the local second-order surface geometry at a regular point, represented for example by the pair (g, B) modulo the gauge identifications of Section 7.5. Let $\mathcal{C}_2$ denote the class of nondegenerate surface-adapted scaled CCS data modulo the same gauge identifications. Then the map

$$\Phi : \mathcal{C}_2 \rightarrow \mathcal{G}_2, \quad \Phi(C) = (g, B),$$

is well defined and locally injective. In particular, each CCS gauge-equivalence class determines a unique local second-order geometric class.

Moreover, in the aligned gauge, or more generally in any fixed orthogonal adapted gauge together with its tangent basis map P , every local second-order geometric class in $\mathcal{G}_2$ admits a representative in $\mathcal{C}_2$. Hence, after gauge fixing, scaled adapted CCS provides a local representation layer for second-order surface geometry.

Proof. Well-definedness follows because the gauge identifications of Section 7.5 do not alter the local geometric class represented by (g, B) ; they only change its coordinates or orientation conventions. The gauge-transport proposition above shows that moving between aligned and orthogonal adapted gauges preserves the geometric invariants. Local injectivity is exactly the content of Theorem 4: once the gauge is fixed, the CCS data determine g , B , S , and the curvature invariants uniquely, so two CCS fields with the same gauge-fixed image in $\mathcal{G}_2$ are locally second-order equivalent. Local existence is provided constructively by the preceding lemma. Therefore Φ is a local representation map in the stated sense. $\square$

7.11. Corollary: CCS as a complete local second-order representation layer

Corollary 9. After gauge fixing, scaled adapted CCS is not only a recoverability device but a complete local representation layer for regular second-order surface geometry in the main theorem setting of this paper.

Reason. Theorem 4 gives local completeness on the recovery side, while Theorem 5 upgrades that statement to a representation theorem by adding the local existence of CCS representatives for the admissible second-order geometric classes. $\square$

7.12. What is and is not claimed

The theorem claims local second-order completeness for the geometric layer. It does **not** claim that every conceivable higher-order field, non-orthogonal structure, or physical coupling is already formalized inside the current theorem chain.

8. Discrete CCS and Second-Order Consistency

8.1. Discrete tangent-scale frame

Let a grid of step size h be given on the parameter domain, and let

$$C_h = (o_h, R_h, D_h)$$

be a discrete CCS sampled on grid points. Define

$$T_h = [s_{1,h}e_{1,h} \quad s_{2,h}e_{2,h}], \quad g_h = T_h^\top T_h.$$

Using centered differences,

$$\delta_u f = \frac{f(u+h, v) - f(u-h, v)}{2h}, \quad \delta_v f = \frac{f(u, v+h) - f(u, v-h)}{2h},$$

define the discrete induced connection by

$$\Gamma_{u,h} = g_h^{-1} T_h^\top (\delta_u T_h), \quad \Gamma_{v,h} = g_h^{-1} T_h^\top (\delta_v T_h).$$

Discrete B_h , S_h , K_h , and H_h are then defined by the same recovery chain.

8.2. Lemma: Discrete normal consistency

Lemma 6. Under the regularity and nondegeneracy assumptions of Theorem 6, assume that the discrete tangent vectors and their centered differences are second-order consistent in the fixed gauge of Theorem 4. Then the discrete unit normal n_h recovered from the discrete tangent-scale frame satisfies

$$n_h = n + O(h^2).$$

Proof. Let the continuous tangent vectors be t_1, t_2 and the discrete approximations be $t_{1,h}, t_{2,h}$ with

$$t_{a,h} = t_a + O(h^2).$$

Then the cross product is bilinear, so

$$t_{1,h} \times t_{2,h} = t_1 \times t_2 + O(h^2).$$

Because the surface patch is regular and the tangent-scale frame is uniformly nondegenerate, $\|t_1 \times t_2\|$ is bounded away from zero. Normalization is smooth on this set, hence the normalized discrete normal also satisfies $n_h = n + O(h^2)$. $\square$

8.3. Lemma: Discrete second fundamental form consistency in a fixed gauge

Lemma 7. Under the assumptions of Lemma 6, assume further that the centered differences of the discrete tangent-scale vectors are second-order consistent. Then the second fundamental form recovered in the fixed CCS tangent gauge satisfies

$$\widetilde{B}_h = \widetilde{B} + O(h^2).$$

Proof. By hypothesis,

$$\delta_\mu \tau_{a,h} = \partial_\mu \tau_a + O(h^2),$$

and by Lemma 6,

$$n_h = n + O(h^2).$$

Therefore each recovered coefficient satisfies

$$\langle \delta_\mu \tau_{a,h}, n_h \rangle = \langle \partial_\mu \tau_a, n \rangle + O(h^2).$$

Since $\widetilde{B}_h$ is assembled from finitely many such coefficients in the fixed gauge, the whole matrix satisfies $\widetilde{B}_h = \widetilde{B} + O(h^2)$. $\square$

8.4. Theorem 6: Discrete consistency of the CCS recovery chain

Theorem 6. Let the continuous CCS field $C = (o, R, D)$ be of class C^4 , and let the discrete CCS field $C_h = (o_h, R_h, D_h)$ satisfy

$$o_h = o + O(h^2), \quad R_h = R + O(h^2), \quad D_h = D + O(h^2)$$

at grid points, with uniform nondegeneracy of the tangent-scale metric. Assume in addition that the discrete normal field and the discrete normal components of differentiated tangent-scale vectors are computed by second-order consistent centered schemes in the same fixed gauge used in Theorem 4. Then:

$$\begin{aligned} \Gamma_{\mu,h} &= \Gamma_\mu + O(h^2), & \mu &\in \{u, v\}, \\ B_h &= B + O(h^2), \\ S_h &= S + O(h^2), \\ K_h &= K + O(h^2), & H_h &= H + O(h^2). \end{aligned}$$

Proof. Since $R_h = R + O(h^2)$ and $D_h = D + O(h^2)$, the tangent-scale frame satisfies

$$T_h = T + O(h^2).$$

Hence

$$g_h = g + O(h^2), \quad g_h^{-1} = g^{-1} + O(h^2),$$

by smooth inversion on the nondegenerate set. Standard truncation estimates for centered differences on C^4 functions give

$$\delta_u T_h = \partial_u T + O(h^2), \quad \delta_v T_h = \partial_v T + O(h^2).$$

Substituting into the definition of $\Gamma_{\mu,h}$ yields the $O(h^2)$ consistency of the induced connection. Lemma 6 gives $n_h = n + O(h^2)$, and Lemma 7 gives $\widetilde{B}_h = \widetilde{B} + O(h^2)$ in the fixed tangent gauge. By Lemma 4, the same basis transformation as in Theorem 4 yields $B_h = B + O(h^2)$. Then

$$S_h = g_h^{-1} B_h = S + O(h^2),$$

and the smoothness of determinant and trace gives the curvature estimates. $\square$

8.5. Consequence

Theorem 6 is the rigorous bridge from the continuous theory to an actual discrete algorithm. It shows that CCS is not merely a symbolic reformulation: the entire recovery chain remains second-order consistent when the coordinate-system object itself is approximated to second order.

9. Numerical Interpretation and Why CCS Can Be Stable

9.1. Implementation alignment

The current `coordsystem` Python library already reflects the basic computational realization of this theory.

- `coord3` is the software-level CCS object.

- `IntrinsicGradientOperator` builds and compares local adapted coordinate objects.
- `compute_connection_matrices` implements the connection-style comparison

$$G_i \approx \frac{C(u+h) - C(u-h)}{2h} C(u)^{-1}.$$

- `compute_all_curvatures` recovers g, B, S, K, H , principal curvatures, and related invariants.

This correspondence matters because it shows that the paper is not introducing an abstract theorem chain disconnected from the actual computational path. At the library level, the CCS claim is visible in three distinct ways.

First, `coord3` is not only a passive storage container. It supports object-level multiplication and division semantics, so local coordinate systems participate directly in computation. Second, `IntrinsicGradientOperator` computes local adapted frames and compares nearby coordinate objects rather than only differentiating raw point samples. Third, `compute_connection_matrices` and `compute_all_curvatures` realize the full pipeline

$$\text{CCS object} \rightarrow \text{local variation} \rightarrow \text{induced geometry} \rightarrow \text{curvature invariants},$$

which is exactly the object-level reformulation claimed in Sections 1 and 10.

In other words, the software architecture is already aligned with the mathematical thesis: the discrete approximation target is the coordinate-system object itself, not merely an intermediate derivative stencil on positions.

9.2. Equal-input equal-output principle

A fair numerical comparison must use:

1. the same primitive input data,
2. the same target output package.

For curvature comparison, the meaningful output package is not only a single scalar such as K . A structurally complete package is

$$\{g, B, S, K, H, k_1, k_2, R_{1212}\}.$$

Only under such matched conditions do comparisons of accuracy, sampling cost, and runtime have geometric meaning.

9.3. A local perturbation proposition

The experiments suggest that CCS is more robust on poor-quality discrete data. A full general robustness theorem is not yet available, but the following local proposition explains why the advantage is plausible.

Proposition 6. Assume that the discrete tangent-scale frame satisfies

$$T_h = T + \Delta T, \quad \|\Delta T\| = O(\varepsilon),$$

with nondegenerate metric and bounded first derivatives. Then the induced connection error satisfies

$$\Gamma_{\mu,h} - \Gamma_\mu = O(\varepsilon) + O(h^2),$$

provided the discrete derivatives of T_h are computed by a second-order centered scheme and the perturbation field is first-order bounded.

Reason. The formula

$$\Gamma_{\mu,h} = g_h^{-1} T_h^\top (\delta_\mu T_h)$$

depends smoothly on T_h , g_h^{-1} , and $\delta_\mu T_h$ on the nondegenerate set. Hence the error propagates through matrix multiplication and inversion at first order in the tangent-frame perturbation. By contrast, purely position-based second-fundamental-form routes often expose curvature directly to second differences of noisy positions or normals, which can amplify local sampling distortion more strongly.

This is not yet a full mesh theorem, but it gives a mathematically coherent explanation for the observed robustness trend.

10. Why CCS Is More Than an Algorithmic Improvement

10.1. Classical pipeline versus CCS pipeline

A classical local-curvature pipeline is typically organized as

surface sample $\rightarrow$ position derivatives $\rightarrow$ fundamental forms $\rightarrow$ curvature.

The CCS pipeline is organized as

surface sample $\rightarrow$ local coordinate-system object $\rightarrow$ object variation $\rightarrow$ induced geometry.

This is not a cosmetic rearrangement. In the first route, geometry is reconstructed from derivatives of the embedding. In the second route, geometry is recovered from the variation law of a structured local object that already encodes directional and metric information.

10.2. Proposition 7: Object-level reformulation criterion

Proposition 7. Consider a framework for local surface geometry. Suppose that:

1. its primary local state is a computable object field $\mathcal{O}$ rather than only raw positional samples,
2. its differential law is defined directly on $\mathcal{O}$,
3. its geometric invariants are recovered by canonical projection from the differential law of $\mathcal{O}$,
4. its discrete algorithm approximates $\mathcal{O}$ itself and preserves the invariant-recovery chain consistently.

Then the framework is not merely an algorithmic variant inside a fixed paradigm; it constitutes an object-level reformulation of the problem class.

Proof. If the primary state, the differentiation rule, the invariant-recovery mechanism, and the discrete approximation target are all changed at once, then the formal representation of the problem has changed. That is more than replacing one numerical approximation step inside a fixed representation. It is an object-level reformulation. $\square$

10.3. Corollary 8: CCS satisfies the object-level reformulation criterion

Corollary 8. In the scaled adapted setting developed in this paper, CCS satisfies the criterion of Proposition 7.

Reason.

1. The primary local state is the coordinate-system object $C = (o, s, R)$.
2. The differential law is given by the local variation G_μ together with the induced geometric projection Γ_μ .
3. The invariants g , B , S , K , H , and k_i are recovered canonically from this structure.
4. Theorem 6 shows that the discrete method approximates the whole recovery chain by approximating CCS itself to second order.

Therefore CCS is stronger than a new curvature estimator: it is an object-level reformulation of local geometric computation.

10.4. The real ceiling of the claim

Corollary 8 justifies the following strong statement:

CCS is a paradigm-level innovation within the problem class of computable local geometric representation and recovery.

But it does not justify the much broader slogan that CCS has already overturned all of differential geometry or all of geometry processing. That larger statement would require:

1. a mature non-orthogonal extension,
2. a more general irregular-mesh and point-cloud theory,
3. systematic replacement results across multiple established pipelines,
4. broader external validation.

11. Numerical Evidence

11.1. Parametric-surface benchmarks

Existing benchmark studies on analytic surfaces show the following consistent picture.

1. The CCS route achieves accurate recovery of Gaussian and mean curvature.
2. The convergence behavior is compatible with second-order consistency in the tested settings.
3. Under equal-input equal-output comparisons, CCS remains competitive in sampling efficiency and runtime.

These results support the continuous-to-discrete theorem chain rather than replacing it.

11.2. Low-quality mesh robustness

The supplemental low-quality-mesh experiments provide the strongest currently available empirical evidence for numerical advantage in the discrete setting. Across coarse meshes, noisy normals, tangent perturbations, and anisotropic skinny sampling, the reported aggregate improvements are:

- average $9.13\times$ reduction in the 95th-percentile Gaussian-curvature error,
- average $3.06\times$ reduction in the 95th-percentile mean-curvature error,
- Gaussian-curvature sign accuracy improved from about 73.8% to about 89.4%.

These are significant results. They justify the statement that, in the tested benchmark regimes, the CCS route is numerically more robust than the compared classical route.

11.3. Disciplined interpretation

The numerical evidence supports the following and only the following claims.

1. CCS is computationally effective in the studied benchmark settings.
2. CCS shows stronger robustness on low-quality discrete geometry in those settings.
3. The robustness trend is structurally consistent with the induced-connection viewpoint.

The evidence does **not** yet justify a theorem of universal superiority over all mesh or point-cloud methods.

12. Boundaries and Resolved Overclaims

This paper intentionally sharpens the claims around CCS.

12.1. Full right Maurer-Cartan form

The full right-trivialized variation of the ambient CCS object is a pure-gauge object at the group level. It does not directly equal nonzero surface curvature.

12.2. Pure commutator curvature

The commutator term alone is not the curvature in general. The correct exact formula is

$$\Omega_{uv}^\Gamma = \partial_u \Gamma_v - \partial_v \Gamma_u + [\Gamma_u, \Gamma_v].$$

12.3. General mesh / point-cloud theory

A full general theory for arbitrary meshes or point clouds has not yet been completed. What currently exists is a strong theorem chain for smooth surfaces plus discrete consistency, together with encouraging but still benchmark-bound experiments.

12.4. Blanket complexity declarations

Without a precise computational model, data-access model, and output specification, broad asymptotic complexity claims are not rigorous. Complexity should be discussed only relative to clearly specified algorithms and output packages.

13. Main Conclusion

The strongest correct conclusion is not that CCS is “just another curvature algorithm.” The stronger conclusion is this:

CCS is a coordinate-system object theory designed for the differential problem of spatial vectors under simultaneous variation of position, orientation, and scale.

Within that theory:

1. the local variation splits exactly into connection, bending, and scale layers,
2. a genuine surface connection is induced by tangential metric projection,
3. the exact curvature formula holds at the induced-connection level,

4. scaled adapted CCS provides a complete local representation layer for regular second-order surface geometry once the tangent gauge and basis transformation are made explicit,
5. the discrete recovery chain is second-order consistent once the normal recovery and second fundamental form recovery are required to be second-order consistent in the same fixed gauge,
6. the numerical implementation has a coherent theoretical explanation and experimentally verified advantages on difficult discrete data.

That is already a substantial theorem-level foundation. It is also the right foundation on which later non-orthogonal generalizations, internal-complex extensions, and physics-facing layers can be built.

14. Next Steps

The natural next steps are:

1. formalize the non-orthogonal extension as a separate theorem layer,
2. isolate the discrete normal-recovery and second-fundamental-form consistency lemmas in full detail,
3. prove a sharper local noise-propagation theorem comparing the CCS route with classical second-difference routes,
4. only after the geometric foundation is complete, move to the higher physical narrative.

Appendix A. Core theorem list

1. Theorem 1: exact CCS differential splitting.
2. Theorem 2: natural projection from CCS to the induced geometric connection.
3. Theorem 3: exact curvature formula for the induced connection.
4. Lemma 4: basis-covariant recovery of the second fundamental form.
5. Gauge-transport proposition between aligned and orthogonal adapted gauges.
6. Theorem 4: local second-order completeness of scaled adapted CCS.
7. Constructive local realization lemma for a second-order geometric class.
8. Theorem 5: local representation theorem for scaled adapted CCS.
9. Corollary 9: CCS as a complete local second-order representation layer.
10. Lemma 6: discrete normal consistency.
11. Lemma 7: discrete second fundamental form consistency in a fixed gauge.
12. Theorem 6: discrete consistency of the CCS recovery chain.
13. Proposition 6: local perturbation explanation for numerical robustness.
14. Proposition 7: object-level reformulation criterion.
15. Corollary 8: CCS satisfies the object-level reformulation criterion.