

# Computable Coordinate Systems: From Quaternions to Object-Level Spatial Calculus

Quaternionic origin, object algebra, differentiation, and  
ordered globalization

A coordinate system is promoted from passive background to a local spatial measuring object. Its logarithmic relative variation is the differential datum of space; its ordered composition is the integral reconstruction of space.

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## Abstract

We introduce the **computable coordinate system** (`coord`) as a first-class mathematical object for spatial calculus. A three-dimensional `coord` carries an origin, an orientation, and three positive local ruler lengths,

$$\mathcal{C} = (o, q, s) \equiv (o, R, D), \quad E = RD, \quad D = \text{diag}(s_1, s_2, s_3),$$

where  $o \in \mathbb{R}^3$ ,  $q \in S^3 = \text{Spin}(3)$ ,  $R = \rho(q) \in SO(3)$ , and  $s \in \mathbb{R}_{>0}^3$ . The object is derived naturally from quaternion geometry. The imaginary quaternions provide oriented Euclidean three-space; unit quaternions generate oriented orthonormal frames; nonzero quaternions generate rotations together with uniform scale; affine localization adds the origin; and independent local ruler lengths produce the scaled frame underlying `coord`.

The resulting theory treats spatial differentiation as **localization by relative coordinate objects** and spatial integration as **globalization by ordered composition**. Multiplication composes finite coordinate transformations, division extracts relative frames, logarithmic subtraction maps finite differences into a tangent algebra, and exponential addition reconstructs finite objects from infinitesimal increments. For a coordinate-system field  $\mathcal{C}(x)$ , the logarithmic derivative

$$G_\mu = (\partial_\mu \widehat{\mathcal{C}}) \widehat{\mathcal{C}}^{-1}$$

splits exactly into translational variation, rotational connection, and scale evolution. Conversely, a compatible infinitesimal field reconstructs  $\mathcal{C}$  through a product integral or path-ordered exponential. Closed-loop composition yields holonomy and provides the natural passage from local differential geometry to global geometric and topological information.

For surface-adapted fields, the full ambient logarithmic derivative is a pure-gauge quantity and does not by itself equal surface curvature. The genuine tangent connection is obtained by metric-adapted projection,

$$\Gamma_\mu = g^{-1} T^\top (\partial_\mu T),$$

and its curvature is

$$\Omega_{uv}^\Gamma = \partial_u \Gamma_v - \partial_v \Gamma_u + [\Gamma_u, \Gamma_v].$$

The same object recovers the first and second fundamental forms, shape operator, Gaussian curvature, mean curvature, and principal curvatures. Thus `coord` is not proposed as a new number system, nor merely as a reformulation of a curvature estimator. It is an object theory in which local spatial measurement, differential comparison, integral reconstruction, and geometric computation are expressed within one coherent algebraic language.

**Keywords:** computable coordinate system, quaternion, frame field, spatial calculus, object algebra, logarithmic derivative, product integral, connection, curvature, holonomy

## 1. Introduction

### 1.1. The problem of spatial calculus

Ordinary calculus begins with scalar or vector subtraction. For a function  $f$  on a linear space,

$$\partial_\mu f(x) = \lim_{h \rightarrow 0} \frac{f(x + h e_\mu) - f(x)}{h}.$$

This expression is meaningful because the values of  $f$  belong to a common linear space. A spatial frame is different. Two frames may have different origins, orientations, and local units. Their componentwise difference is generally neither an intrinsic frame nor a coordinate-independent geometric increment.

The fundamental question addressed here is:

*How should one differentiate and integrate spatial objects when position, orientation, and scale vary simultaneously?*

The proposed answer is to promote the local coordinate system itself to the primary computational object. Nearby spatial states are first compared by a relative coordinate transformation. This finite relative transformation is then linearized by a logarithm. Conversely, infinitesimal coordinate changes are integrated by ordered composition. The resulting calculus has the schematic form

|                                                                                                                                                                           |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| global coordinate object $\xrightarrow{\text{relative localization}}$ infinitesimal spatial change $\xrightarrow{\text{ordered globalization}}$ global coordinate object. |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

### 1.2. Main idea

A three-dimensional computable coordinate system is represented by

$$C = (o, q, s),$$

where  $o$  is an affine origin,  $q$  is a unit quaternion representing orientation, and  $s = (s_1, s_2, s_3)$  gives the positive lengths of three orthogonal local rulers. Equivalently,

$$C = (o, R, D), \quad R \in SO(3), \quad D = \text{diag}(s_1, s_2, s_3).$$

The associated local-to-world map is

$$\Phi_C(x) = o + RDx.$$

This simple action already contains the three operations required by spatial measurement:

1.  $o$  locates the local space;
2.  $R$  orients the local space;
3.  $D$  determines its local units.

### 1.3. Contributions

This paper establishes the following results.

1. It derives the oriented orthogonal `coord` object canonically from quaternionic three-space and affine measurement axioms.
2. It proves a representation and uniqueness theorem for oriented orthogonal affine measuring frames.
3. It distinguishes the exact similarity group, the orthogonal anisotropic `coord` object space, and the full affine closure in  $GL^+(3)$ .
4. It defines a typed coordinate-object algebra for composition, relative localization, logarithmic subtraction, and exponential reconstruction.
5. It derives the differential splitting of a smooth coordinate-system field into translation, rotation, and scale.
6. It proves the local inverse relation between logarithmic differentiation and product integration.
7. It identifies compatibility, path dependence, holonomy, and curvature as successive local-to-global consequences of the same object calculus.
8. It derives surface metric, connection, second fundamental form, and curvature from a surface-adapted `coord` field.

## 2. Quaternionic Origin of Three-Dimensional Frames

### 2.1. Imaginary quaternions as oriented Euclidean space

Let

$$\mathbb{H} = \{a + xi + yj + zk : a, x, y, z \in \mathbb{R}\}$$

be the quaternion algebra, with

$$i^2 = j^2 = k^2 = ijk = -1.$$

Its imaginary subspace is

$$\mathbb{V} := \text{Im } \mathbb{H} = \text{span}_{\mathbb{R}} \{i, j, k\} \cong \mathbb{R}^3.$$

For  $u, v \in \mathbb{V}$ , quaternion multiplication decomposes as

$$uv = -\langle u, v \rangle + u \times v.$$

**Proposition 1 (Quaternionic encoding of Euclidean geometry).** The scalar and imaginary parts of  $uv$  recover the Euclidean inner product and oriented cross product:

$$\langle u, v \rangle = -\text{Re}(uv), \quad u \times v = \text{Im}(uv).$$

Consequently, orthogonality, norm, orientation, and oriented area are intrinsic to quaternion multiplication on  $\mathbb{V}$ .

*Proof.* Write

$$u = u_1 i + u_2 j + u_3 k, \quad v = v_1 i + v_2 j + v_3 k.$$

Expanding with the quaternion multiplication table gives scalar part

$$-(u_1 v_1 + u_2 v_2 + u_3 v_3)$$

and imaginary part equal to the standard determinant expression for  $u \times v$ .  $\square$

Thus the space on which a `coord` acts is not attached externally to the quaternions. It is already present as  $\text{Im } \mathbb{H}$ .

## 2.2. Unit quaternions and oriented frames

Let

$$S^3 = \{q \in \mathbb{H} : |q| = 1\}.$$

For  $q \in S^3$ , define

$$\rho_q(v) = qv\bar{q}, \quad v \in \mathbb{V}.$$

**Proposition 2 (Quaternionic frame action).** The map  $\rho_q$  belongs to  $SO(3)$ , the assignment

$$\rho : S^3 \rightarrow SO(3)$$

is a surjective group homomorphism, and

$$\ker \rho = \{1, -1\}.$$

This is the classical double covering

$$\text{Spin}(3) \cong S^3 \rightarrow SO(3).$$

Apply  $\rho_q$  to the canonical imaginary basis:

$$e_1 = qi\bar{q}, \quad e_2 = qj\bar{q}, \quad e_3 = qk\bar{q}.$$

Then  $(e_1, e_2, e_3)$  is a positively oriented orthonormal frame. Its matrix is

$$R(q) = [e_1 \mid e_2 \mid e_3] \in SO(3).$$

**Corollary 1 (Natural origin of the unit coordinate object).** An oriented orthonormal three-frame is equivalent to a unit quaternion modulo sign:

$$\text{UCoord}^+(3) \cong S^3 / \{\pm 1\} \cong SO(3).$$

The sign ambiguity  $q \sim -q$  is a gauge redundancy, not a geometric ambiguity.

## 2.3. Nonzero quaternions and uniform scale

Let  $p \in \mathbb{H}^\times = \mathbb{H} \setminus \{0\}$  and write

$$p = rq, \quad r = |p| > 0, \quad q \in S^3.$$

Define

$$L_p(v) = pv\bar{p}.$$

Then

$$L_p(v) = r^2 qv\bar{q} = \lambda R(q)v, \quad \lambda = r^2.$$

**Proposition 3 (Quaternionic similarity action).** A nonzero quaternion determines an orientation-preserving linear similarity of  $\mathbb{V}$ . Moreover,

$$L_p \circ L_q = L_{pq},$$

and therefore

$$\mathbb{H}^\times / \{\pm 1\} \cong \mathbb{R}_{>0} \times SO(3).$$

**Proof.** Using  $\overline{p\bar{q}} = \bar{q}\bar{p}$ ,

$$L_p(L_q(v)) = p(qv\bar{q})\bar{p} = (pq)v\overline{p\bar{q}} = L_{pq}(v).$$

The norm is multiplicative, so the induced scale is multiplicative. The only elements acting trivially are 1 and  $-1$  after scale is fixed.  $\square$

This proposition separates two quaternionic actions that must not be conflated:

$$pv p^{-1} = R(q)v$$

removes the norm and produces pure rotation, whereas

$$pv\bar{p} = |p|^2 R(q)v$$

retains uniform scale.

### 3. Natural Completion to the `coord` Object

#### 3.1. Affine localization

Vectors do not possess an intrinsic position. A coordinate system must therefore act on an affine space modeled on  $\mathbb{V}$ . Introduce an origin  $o \in \mathbb{V}$  and define

$$\Phi_{(o,p)}(x) = o + px\bar{p}.$$

The pair  $(o, p)$  represents a translated, rotated, uniformly scaled local space.

If  $\Phi_2 \circ \Phi_1$  denotes ordinary function composition, then

$$\Phi_{(o_2,p_2)} \circ \Phi_{(o_1,p_1)} = \Phi_{(o_2 + L_{p_2}(o_1), p_2 p_1)}.$$

Thus affine localization produces the orientation-preserving similarity group

$$\text{Sim}^+(3) = \mathbb{R}^3 \rtimes (\mathbb{R}_{>0} \times SO(3)).$$

#### 3.2. Independent local rulers

A single quaternion supplies only one scale. Spatial calculus on curves, surfaces, and anisotropic geometric data requires three independently measurable local ruler lengths. Let

$$s = (s_1, s_2, s_3) \in \mathbb{R}_{>0}^3, \quad D_s = \text{diag}(s_1, s_2, s_3).$$

For

$$x = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k},$$

define

$$D_s x = s_1 x_1 \mathbf{i} + s_2 x_2 \mathbf{j} + s_3 x_3 \mathbf{k}.$$

With  $q \in S^3$ , define

$$\Phi_C(x) = o + q(D_s x)\bar{q}.$$

In matrix notation,

$$\Phi_C(x) = o + R(q)D_s x.$$

**Definition 1 (Computable coordinate system).** An oriented orthogonal three-dimensional computable coordinate system is

$$C = (o, q, s) \equiv (o, R, D),$$

where

$$o \in \mathbb{R}^3, \quad [q] \in S^3/\{\pm 1\}, \quad R = \rho(q) \in SO(3), \quad s \in \mathbb{R}_{>0}^3.$$

Its scaled frame matrix is

$$E = RD = [\tau_1 \mid \tau_2 \mid \tau_3], \quad \tau_a = s_a e_a.$$

The corresponding object space is

$$C_{\perp}^+(3) = \mathbb{R}^3 \times SO(3) \times \mathbb{R}_{>0}^3.$$

### 3.3. Representation theorem

**Theorem 1 (Quaternionic representation of oriented orthogonal affine frames).** Let  $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be an affine map such that the three vectors

$$v_a = \Phi(e_a) - \Phi(0), \quad a = 1, 2, 3,$$

are nonzero, mutually orthogonal, and positively oriented. Then there exist unique

$$o \in \mathbb{R}^3, \quad s \in \mathbb{R}_{>0}^3, \quad [q] \in S^3 / \{\pm 1\},$$

such that

$$\Phi(x) = o + q(D_s x) \bar{q}.$$

*Proof.* Set

$$o = \Phi(0), \quad s_a = \|v_a\|, \quad e_a = \frac{v_a}{s_a}.$$

The hypotheses imply that  $(e_1, e_2, e_3)$  is a positively oriented orthonormal frame, so

$$R = [e_1 \mid e_2 \mid e_3] \in SO(3).$$

The double covering  $S^3 \rightarrow SO(3)$  provides exactly two unit quaternions  $q$  and  $-q$  representing  $R$ . Hence  $[q]$  is unique. The ruler lengths are uniquely determined by the norms of the affine axis vectors, and the origin is uniquely  $\Phi(0)$ . Therefore

$$\Phi(x) = o + RD_s x = o + q(D_s x) \bar{q}.$$

□

This theorem gives a precise sense in which `coord` is naturally derived from quaternions. Quaternion geometry provides the oriented frame; affine measurement supplies position; independent axis norms supply the local rulers.

### 3.4. Volumetric scale and anisotropic shape

The three ruler lengths admit the canonical decomposition

$$\lambda = (s_1 s_2 s_3)^{1/3}, \quad a_i = \frac{s_i}{\lambda}, \quad a_1 a_2 a_3 = 1.$$

Thus

$$D_s = \lambda D_a.$$

The scalar  $\lambda$  is the volumetric scale already compatible with the norm of a nonzero quaternion. The unimodular diagonal matrix  $D_a$  contains two additional anisotropic shape degrees of freedom. Hence the degrees of freedom separate as

$$\underbrace{3}_{\bar{o}} + \underbrace{3}_{q/\{\pm 1\}} + \underbrace{1}_{\bar{\lambda}} + \underbrace{2}_{\bar{D}_a} = 9.$$

### 3.5. Handedness

Unit quaternions cover  $SO(3)$  and therefore represent orientation-preserving frames. A theory including reflections must add a discrete handedness variable or pass from  $Spin(3)$  to an appropriate  $Pin(3)$  structure. This is an extension of the present oriented theory, not a degree of freedom hidden inside a unit quaternion.

## 4. Algebra of Coordinate Objects

### 4.1. Homogeneous representation

Associate to  $C = (o, E)$  the homogeneous matrix

$$\widehat{C} = \begin{pmatrix} E & o \\ 0^\top & 1 \end{pmatrix}.$$

It acts on a local point  $x$  by

$$\begin{pmatrix} \Phi_C(x) \\ 1 \end{pmatrix} = \widehat{C} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

The full orientation-preserving affine group is

$$\text{Aff}^+(3) = \mathbb{R}^3 \rtimes GL^+(3).$$

Within this affine envelope, composition and inversion are exact:

$$(o_2, E_2) \circ (o_1, E_1) = (o_2 + E_2 o_1, E_2 E_1),$$

and

$$(o, E)^{-1} = (-E^{-1}o, E^{-1}).$$

### 4.2. Multiplication and division

Fix the convention

$$C_2 * C_1 \iff \widehat{C}_2 \widehat{C}_1,$$

so that  $C_1$  acts first and  $C_2$  acts second. Software libraries may reverse the written order to follow child-parent syntax; the mathematics is unchanged once one convention is fixed.

**Definition 2 (Composition).** Coordinate multiplication is finite spatial composition:

$$C_2 * C_1 = (o_2 + E_2 o_1, E_2 E_1).$$

**Definition 3 (Relative coordinate object).** The right-relative and left-relative coordinate objects are

$$C_2 / C_1 := C_2 * C_1^{-1},$$

and

$$C_1 \setminus C_2 := C_1^{-1} * C_2,$$

respectively. The first expresses the change in a spatial or world-trivialized convention; the second expresses it in a body or local-trivialized convention.

For a world point  $p$ , localization into  $C$  is

$$x = E^{-1}(p - o),$$

and globalization is

$$p = o + Ex.$$

These are naturally read as point division by  $C$  and point multiplication by  $C$ .

### 4.3. Logarithmic subtraction and exponential addition

Ordinary addition and subtraction are not intrinsic on a nonlinear coordinate-object space. Their geometrically correct replacements are the Lie logarithm and exponential.

**Definition 4 (Object subtraction).** For two sufficiently close affine coordinate objects,

$$C_2 \ominus C_1 := \text{Log}(\widehat{C}_2 \widehat{C}_1^{-1}).$$

The result belongs to the affine Lie algebra

$$\mathfrak{aff}(3) = \left\{ \begin{pmatrix} B & v \\ 0^\top & 0 \end{pmatrix} : B \in \mathfrak{gl}(3), v \in \mathbb{R}^3 \right\}.$$

**Definition 5 (Object addition).** For an infinitesimal increment  $\zeta \in \mathfrak{aff}(3)$ ,

$$\zeta \oplus C := \text{Exp}(\zeta) \widehat{C}.$$

Where defined,

$$(C_2 \ominus C_1) \oplus C_1 = C_2.$$

Thus the four object operations have distinct types:

- $*$  : finite composition,
- $/$  : relative localization,
- $\ominus$  : finite object difference mapped to an infinitesimal algebra,
- $\oplus$  : infinitesimal increment reconstructed as a finite object.

This typed interpretation preserves the intuitive four-operation language without treating a nonlinear frame space as an ordinary vector space.

### 4.4. Exact closure and the anisotropic boundary

Uniformly scaled frames,

$$E = \lambda R,$$

are closed under multiplication and form  $\text{Sim}^+(3)$ . Orthogonal anisotropically scaled matrices,

$$\mathcal{E}_\perp = \{RD : R \in SO(3), D \text{ positive diagonal}\},$$

are not closed under arbitrary multiplication.

**Theorem 2 (Closure criterion for orthogonal anisotropic frames).** Let

$$E_1 = R_1 D_1, \quad E_2 = R_2 D_2.$$

Then  $E_2 E_1 \in \mathcal{E}_\perp$  if and only if

$$R_1^\top D_2^2 R_1$$

is diagonal.

**Proof.** A matrix belongs to  $\mathcal{E}_1$  exactly when its columns are mutually orthogonal, equivalently when  $E^\top E$  is diagonal positive. For  $E = E_2 E_1$ ,

$$E^\top E = D_1 R_1^\top D_2 R_2^\top R_2 D_2 R_1 D_1 = D_1 R_1^\top D_2^2 R_1 D_1.$$

Since  $D_1$  is positive diagonal, this matrix is diagonal exactly when  $R_1^\top D_2^2 R_1$  is diagonal.  $\square$

When the entries of  $D_2$  are distinct, the allowed generic rotations are restricted to signed axis permutations. Arbitrary rotation-scale composition produces shear.

**Corollary 2.** The space  $C_\perp^+(3)$  is a natural manifold of orthogonal measuring frames, but it is not a subgroup of  $\text{Aff}^+(3)$  under general anisotropic composition.

There are three mathematically valid regimes:

1. use  $\text{Sim}^+(3)$  when scale is uniform;
2. use  $\text{Aff}^+(3)$  or  $GL^+(3)$  when exact arbitrary composition is required;
3. retain  $C_\perp^+(3)$  as an adapted measuring-frame space and specify an explicit refactorization or retraction after operations that generate shear.

This distinction is essential for a rigorous coord algebra.

## 5. Coordinate-System Fields and Spatial Differentiation

### 5.1. The coordinate-system field

Let  $M$  be a smooth parameter manifold. A computable coordinate-system field is a smooth map

$$C : M \longrightarrow C_\perp^+(3)$$

or, when affine closure is needed,

$$C : M \longrightarrow \text{Aff}^+(3).$$

In local parameters  $x^\mu$ ,

$$C(x) = (o(x), R(x), D(x)).$$

### 5.2. Left and right logarithmic derivatives

There are two canonical differential comparisons:

$$\Theta_\mu := \widehat{C}^{-1}(\partial_\mu \widehat{C}),$$

and

$$G_\mu := (\partial_\mu \widehat{C})\widehat{C}^{-1}.$$

$\Theta_\mu$  is body-trivialized: it expresses change in the current local frame.  $G_\mu$  is spatially trivialized: it expresses change in the ambient frame. They are related by

$$G_\mu = \widehat{C}\Theta_\mu\widehat{C}^{-1}.$$

The right-trivialized form has block structure

$$G_\mu = \begin{pmatrix} B_\mu & t_\mu \\ 0^\top & 0 \end{pmatrix},$$

where

$$B_\mu = (\partial_\mu E)E^{-1},$$

and

$$t_\mu = \partial_\mu o - B_\mu o.$$

The left-trivialized translation is more directly local:

$$\Theta_\mu = \begin{pmatrix} E^{-1} \partial_\mu E & E^{-1} \partial_\mu o \\ 0^\top & 0 \end{pmatrix}.$$

### 5.3. Quaternionic rotational derivative

Let  $q(x) \in S^3$ . Define the spatial quaternionic angular rate by

$$a_\mu = (\partial_\mu q) \bar{q}.$$

Differentiating  $q\bar{q} = 1$  shows that

$$\text{Re}(a_\mu) = 0,$$

so  $a_\mu \in \text{Im } \mathbb{H}$ . Let

$$\omega_\mu = 2a_\mu.$$

For each frame axis  $e_a = q\mathbf{e}_a\bar{q}$ ,

$$\partial_\mu e_a = a_\mu e_a - e_a a_\mu = \omega_\mu \times e_a.$$

Consequently,

$$A_\mu := (\partial_\mu R)R^\top = [\omega_\mu]_\times \in \mathfrak{so}(3).$$

### 5.4. Exact differential splitting

Define logarithmic scale rates

$$\sigma_{\mu a} = \partial_\mu \log s_a.$$

**Theorem 3 (Exact coord differential splitting).** For  $E = RD$ ,

$$B_\mu = (\partial_\mu E)E^{-1} = A_\mu + S_\mu,$$

where

$$A_\mu = (\partial_\mu R)R^\top \in \mathfrak{so}(3),$$

and

$$S_\mu = R((\partial_\mu D)D^{-1})R^\top = R \text{diag}(\sigma_{\mu 1}, \sigma_{\mu 2}, \sigma_{\mu 3})R^\top \in \text{Sym}(3).$$

The decomposition is unique.

**Proof.** Using  $E^{-1} = D^{-1}R^\top$ ,

$$\begin{aligned} (\partial_\mu E)E^{-1} &= ((\partial_\mu R)D + R(\partial_\mu D))D^{-1}R^\top \\ &= (\partial_\mu R)R^\top + R(\partial_\mu D)D^{-1}R^\top. \end{aligned}$$

The first term is skew-symmetric because  $R \in SO(3)$ . The second is symmetric because  $(\partial_\mu D)D^{-1}$  is diagonal. Uniqueness follows from the unique decomposition of a real matrix into symmetric and skew-symmetric parts.  $\square$

Thus the first-order state of a coord field separates canonically into

translation + rotation + scale evolution.

### 5.5. Differential as infinitesimal relative division

Let  $e_\mu$  denote a parameter direction and define the finite right-relative increment

$$\Delta_\mu^h C(x) = \widehat{C}(x + h e_\mu) \widehat{C}(x)^{-1}.$$

**Proposition 4 (Logarithmic difference limit).** For a  $C^2$  coordinate-system field,

$$\frac{1}{h} \text{Log}(\Delta_\mu^h C(x)) = G_\mu(x) + O(h).$$

*Proof.* Taylor expansion gives

$$\widehat{C}(x + h e_\mu) \widehat{C}(x)^{-1} = I + h G_\mu(x) + O(h^2).$$

Since  $\text{Log}(I + X) = X + O(\|X\|^2)$  near the identity, division by  $h$  yields the result.  $\square$

This proposition formalizes the central localization principle:

$$\text{spatial derivative} = \lim \frac{\text{relative coordinate object}}{\text{parameter increment}}.$$

The division is geometric; the final scalar division occurs only after the relative object has been mapped into a linear tangent algebra.

## 6. Spatial Integration as Ordered Globalization

### 6.1. Reconstruction along a path

Let  $\gamma : [0, 1] \rightarrow M$  be a smooth path and let

$$G(t) = G_\mu(\gamma(t)) \dot{\gamma}^\mu(t).$$

The defining relation

$$G = (\dot{\widehat{C}}) \widehat{C}^{-1}$$

is equivalent to the reconstruction equation

$$\dot{\widehat{C}} = G \widehat{C}.$$

**Theorem 4 (Product-integral reconstruction).** Let  $G : [0, 1] \rightarrow \mathfrak{aff}(3)$  be continuous and let  $\widehat{C}_0 \in \text{Aff}^+(3)$ . There exists a unique solution of

$$\dot{\widehat{C}}(t) = G(t) \widehat{C}(t), \quad \widehat{C}(0) = \widehat{C}_0.$$

It is

$$\widehat{C}(t) = \mathcal{P} \exp \left( \int_0^t G(\tau) d\tau \right) \widehat{C}_0,$$

where  $\mathcal{P}$  denotes path ordering.

The discrete form is

$$\widehat{C}_{k+1} = \text{Exp}(\Delta t G_k) \widehat{C}_k,$$

and therefore

$$\widehat{C}_n = \text{Exp}(\Delta t G_{n-1}) \cdots \text{Exp}(\Delta t G_0) \widehat{C}_0.$$

This is the coordinate-object analogue of integration. Infinitesimal local changes are not added as unrelated vectors; they are composed in their spatial order.

## 6.2. Local fundamental theorem of coord calculus

**Theorem 5 (Differentiation-integration inverse).** Let  $C(t)$  be a  $C^1$  coordinate path and define

$$G(t) = \dot{\widehat{C}}(t)\widehat{C}(t)^{-1}.$$

Then product integration of  $G$  with initial value  $\widehat{C}(0)$  recovers  $\widehat{C}(t)$  uniquely. Conversely, if  $\widehat{C}$  is reconstructed from a continuous  $G$  by Theorem 4, then

$$\dot{\widehat{C}}\widehat{C}^{-1} = G.$$

*Proof.* Both statements follow from uniqueness of the linear matrix differential equation

$$\dot{Y} = GY.$$

□

Hence

$$\boxed{C \xrightarrow{dCC^{-1}} G \xrightarrow{\mathcal{D}\exp f} C}$$

is a genuine local differential-integral duality.

## 6.3. Compatibility in several parameters

For a right logarithmic derivative

$$G_\mu = (\partial_\mu \widehat{C})\widehat{C}^{-1},$$

commuting parameter derivatives imply the right Maurer–Cartan identity

$$\partial_\mu G_\nu - \partial_\nu G_\mu - [G_\mu, G_\nu] = 0.$$

For the left logarithmic derivative

$$\Theta_\mu = \widehat{C}^{-1}\partial_\mu \widehat{C},$$

the corresponding identity is

$$\partial_\mu \Theta_\nu - \partial_\nu \Theta_\mu + [\Theta_\mu, \Theta_\nu] = 0.$$

**Theorem 6 (Local integrability of an affine logarithmic field).** Let  $U$  be simply connected and let  $G_\mu : U \rightarrow \text{aff}(3)$  be smooth. There exists a local coordinate-system field satisfying

$$\partial_\mu \widehat{C} = G_\mu \widehat{C}$$

if and only if

$$\partial_\mu G_\nu - \partial_\nu G_\mu - [G_\mu, G_\nu] = 0.$$

The solution is unique after specifying  $\widehat{C}$  at one point.

This is the precise condition under which local differential data can be globalized without path ambiguity inside a simply connected patch.

## 7. Closed Loops, Holonomy, and Local-to-Global Geometry

### 7.1. Ordered loop composition

For a connection one-form  $\Gamma$  along a closed path  $\gamma$ , define

$$\text{Hol}_\gamma(\Gamma) = \mathcal{P} \exp \oint_\gamma \Gamma.$$

If the ordered product is not the identity, an object transported around the loop does not return with the same orientation or internal state.

For an infinitesimal coordinate rectangle,

$$\Delta C_\square = \Delta C_u \Delta C_v \Delta C_u^{-1} \Delta C_v^{-1},$$

where juxtaposition denotes ordered composition. The Baker–Campbell–Hausdorff formula shows that the leading nontrivial term is of area order and is governed by curvature.

### 7.2. Full-frame flatness versus induced curvature

The full logarithmic derivative of a globally defined ambient coordinate object is pure gauge. Its Maurer–Cartan curvature vanishes identically with the sign convention appropriate to left or right trivialization. Therefore

$$G_\mu = (\partial_\mu \widehat{C}) \widehat{C}^{-1}$$

must not be interpreted directly as a nonzero surface-curvature connection.

Nonzero geometric curvature arises after one of the following operations:

1. projection to a constrained tangent or normal subbundle;
2. restriction to a gauge field not globally representable by a single smooth frame;
3. comparison across nontrivial transition functions;
4. introduction of an independent connection not equal to a pure-gauge logarithmic derivative.

This distinction is central. The ambient `coord` field supplies complete local variation; geometric curvature is an invariant of the appropriate induced connection.

### 7.3. Topological information

Locally, an oriented frame admits a quaternion lift. Globally, the signs of local lifts and the transition maps between frame patches may fail to admit a single consistent choice. Closed-loop transport, transition cocycles, singularities, and obstructions to global framing are therefore natural global data of a coordinate-object field.

The local-to-global hierarchy is

|                                                                                                                                                                               |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\text{relative coord} \longrightarrow \text{connection} \longrightarrow \text{curvature} \longrightarrow \text{holonomy} \longrightarrow \text{global gluing and topology}.$ |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

Curvature and topology are not identical. Curvature is local differential data; topology is encoded in the global compatibility class of local data and their transitions. The `coord` language places both in one computational chain.

## 8. Surface Geometry from an Adapted `coord` Field

### 8.1. Surface-adapted object

Let

$$X : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

be a regular oriented surface patch. A surface-adapted coord field is

$$C(u, v) = (X(u, v), R(u, v), D(u, v)),$$

where

$$R = [e_1 \mid e_2 \mid n] \in SO(3),$$

$e_1, e_2$  span the tangent plane, and  $n = e_1 \times e_2$  is the unit normal.

Define the scaled tangent vectors

$$\tau_1 = s_1 e_1, \quad \tau_2 = s_2 e_2,$$

and

$$T = [\tau_1 \mid \tau_2] \in \mathbb{R}^{3 \times 2}.$$

## 8.2. Metric and tangent projection

The induced metric is already contained in the measuring frame:

$$g = T^\top T.$$

In the orthogonal tangent gauge,

$$g = \begin{pmatrix} s_1^2 & 0 \\ 0 & s_2^2 \end{pmatrix}.$$

The metric left inverse of  $T$  is

$$T_g^\dagger = g^{-1} T^\top.$$

**Definition 6 (Induced tangent connection).** The tangent connection coefficients induced by the coord field are

$$\Gamma_\mu = g^{-1} T^\top (\partial_\mu T), \quad \mu \in \{u, v\}.$$

**Theorem 7 (Canonical tangent-normal splitting).** For each  $\mu$ ,

$$\partial_\mu T = T \Gamma_\mu + n b_\mu^\top,$$

where

$$b_\mu^\top = n^\top (\partial_\mu T).$$

The first term is the metric-adapted tangential variation and the second is the normal bending variation.

**Proof.** The orthogonal projectors onto tangent and normal spaces are

$$P_T = T g^{-1} T^\top, \quad P_N = n n^\top.$$

Since  $P_T + P_N = I$  on the adapted decomposition,

$$\partial_\mu T = P_T \partial_\mu T + P_N \partial_\mu T = T g^{-1} T^\top \partial_\mu T + n n^\top \partial_\mu T.$$

Substituting the definitions gives the result.  $\square$

## 8.3. Curvature of the induced connection

**Theorem 8 (Exact curvature formula).** For commuting local coordinates  $u, v$ , the curvature of the induced tangent connection is

$$\Omega_{uv}^\Gamma = \partial_u \Gamma_v - \partial_v \Gamma_u + [\Gamma_u, \Gamma_v].$$

It represents the tangent-bundle curvature in the basis determined by  $T$ .

In an oriented orthonormal tangent gauge,

$$\Gamma_\mu = \begin{pmatrix} 0 & -\omega_\mu \\ \omega_\mu & 0 \end{pmatrix},$$

and

$$\Omega_{uv}^\Gamma = K \sqrt{\det g} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

up to the chosen orientation and connection-sign convention.

#### 8.4. Second fundamental form and curvature invariants

Assume the aligned gauge

$$\tau_1 = X_u, \quad \tau_2 = X_v.$$

Define

$$\Pi_{\mu a} = \langle n, \partial_\mu \tau_a \rangle.$$

Commutativity of mixed partial derivatives gives

$$\Pi_{12} = \Pi_{21}.$$

The shape operator is

$$\mathcal{S} = g^{-1} \Pi.$$

Therefore

$$K = \det \mathcal{S}, \quad H = \frac{1}{2} \operatorname{tr} \mathcal{S},$$

and the principal curvatures are the eigenvalues of  $\mathcal{S}$ :

$$k_{1,2} = H \pm \sqrt{H^2 - K}.$$

**Theorem 9 (Local second-order completeness).** In an aligned, nondegenerate, surface-adapted gauge, the `coord` field and its first derivatives determine

$$g, \quad \Gamma_\mu, \quad \Pi, \quad \mathcal{S}, \quad K, \quad H, \quad k_1, \quad k_2.$$

Hence the first jet of the adapted `coord` field determines the complete local second-order geometry of the underlying regular surface.

*Proof.*  $T$  is contained in  $\mathcal{C}$ , so  $g = T^\top T$  is known. The derivatives  $\partial_\mu T$  determine  $\Gamma_\mu$  by tangent projection and determine  $\Pi$  by normal projection. The remaining invariants follow algebraically from  $g^{-1} \Pi$ . Since  $T = (X_u, X_v)$  in the aligned gauge,  $\partial_\mu T$  contains the second derivatives of  $X$ .  $\square$

## 9. Differential Operators in the `coord` Language

### 9.1. Scalar gradient

Let  $f : U \rightarrow \mathbb{R}$ . In an aligned surface `coord`,

$$df = (\partial_u f \quad \partial_v f).$$

The ambient tangent gradient is

$$\operatorname{grad} f = T g^{-1} \begin{pmatrix} \partial_u f \\ \partial_v f \end{pmatrix}.$$

Thus the `coord` object performs both metric inversion and globalization from local components to an ambient vector.

## 9.2. Divergence and Laplace–Beltrami operator

For a tangent vector field

$$V = T \begin{pmatrix} V^u \\ V^v \end{pmatrix},$$

the divergence is

$$\operatorname{div} V = \frac{1}{\sqrt{\det g}} \partial_\mu \left( \sqrt{\det g} V^\mu \right).$$

The Laplace–Beltrami operator is

$$\Delta_g f = \frac{1}{\sqrt{\det g}} \partial_\mu \left( \sqrt{\det g} g^{\mu\nu} \partial_\nu f \right).$$

These are classical invariant formulas. The contribution of `coord` is to package  $T$ ,  $g$ , local directions, scale, and coordinate conversion into the primary object from which the operators are computed.

## 9.3. Covariant derivative

If  $V^{\text{loc}}$  denotes the local component column of a tangent field, then

$$\nabla_\mu V^{\text{loc}} = \partial_\mu V^{\text{loc}} + \Gamma_\mu V^{\text{loc}}.$$

Globalization is

$$V = T V^{\text{loc}}.$$

The local/global pair is therefore

$$V^{\text{loc}} = g^{-1} T^\top V, \quad V = T V^{\text{loc}}.$$

This is the differential version of division by and multiplication with the local coordinate object.

# 10. Global Reconstruction of Curves and Surfaces

## 10.1. Curves

Let a moving `coord` along a curve be

$$C(t) = (r(t), R(t), D(t)).$$

Given local translational velocity  $v(t)$ , angular velocity  $\omega(t)$ , and logarithmic scale velocity  $\sigma(t)$ , reconstruction takes the form

$$\dot{r} = E v,$$

$$\dot{R} = R[\omega]_\times$$

for a body-trivialized angular rate, and

$$\dot{s}_a = \sigma_a s_a.$$

Equivalently, the full object is reconstructed by one product integral in its affine envelope.

This representation integrates position and local geometry together. A reconstructed curve may therefore carry tangent, normal, binormal, scale, and additional semantic or residual fields at every point.

## 10.2. Surfaces

Suppose tangent fields  $\tau_1, \tau_2 : U \rightarrow \mathbb{R}^3$  are prescribed and one seeks

$$X_u = \tau_1, \quad X_v = \tau_2.$$

**Theorem 10 (Position reconstruction from a tangent coord).** Let  $U$  be simply connected and let  $\tau_1, \tau_2$  be  $C^1$ . There exists a map  $X : U \rightarrow \mathbb{R}^3$ , unique up to translation, satisfying

$$X_u = \tau_1, \quad X_v = \tau_2$$

if and only if

$$\partial_v \tau_1 = \partial_u \tau_2.$$

*Proof.* The vector-valued one-form

$$\alpha = \tau_1 du + \tau_2 dv$$

is closed exactly when

$$d\alpha = (\partial_u \tau_2 - \partial_v \tau_1) du \wedge dv = 0.$$

The Poincare lemma on a simply connected domain gives  $\alpha = dX$ , with uniqueness up to an additive constant.  $\square$

The theorem is the positional counterpart of the Maurer–Cartan compatibility theorem. The tangent rulers reconstruct position; the connection reconstructs frame orientation; the scale rates reconstruct local units.

## 10.3. Discrete closure residual

On a discrete grid, two paths to the same nominal vertex generally produce two reconstructed objects,  $C_{uv}$  and  $C_{vu}$ . Define the object closure residual by

$$\varepsilon_{cl} = \|\text{Log}(\widehat{C}_{uv} \widehat{C}_{vu}^{-1})\|.$$

This residual simultaneously measures translational, rotational, and linear-frame disagreement once a norm on  $\text{aff}(3)$  is chosen. Separate weighted components may be used when physical units differ.

The closure residual turns global consistency into a directly auditable quantity.

## 11. Discrete coord Calculus

Let  $C_k = C(x_k)$  be sampled coordinate objects.

### 11.1. Relative difference

The first-order logarithmic difference is

$$\xi_k = \frac{1}{h} \text{Log}(\widehat{C}_{k+1} \widehat{C}_k^{-1}).$$

It preserves the nonlinear geometry of finite frame differences before linearization.

### 11.2. Reconstruction

Given increments  $\xi_k$ ,

$$\widehat{C}_{k+1} = \text{Exp}(h\xi_k) \widehat{C}_k.$$

The update remains in the chosen Lie group when  $\tilde{\zeta}_k$  belongs to its Lie algebra. If the state is restricted to the non-closed orthogonal anisotropic submanifold, the update must either be performed in the affine envelope or followed by an explicitly defined orthogonal-scale retraction.

### 11.3. Infinitesimal loop

For small increments  $h\tilde{\zeta}_u$  and  $h\tilde{\zeta}_v$ ,

$$\text{Exp}(h\tilde{\zeta}_u) \text{Exp}(h\tilde{\zeta}_v) \text{Exp}(-h\tilde{\zeta}_u) \text{Exp}(-h\tilde{\zeta}_v) = \text{Exp}(h^2[\tilde{\zeta}_u, \tilde{\zeta}_v] + O(h^3)).$$

When derivative terms are included across a spatial cell, the complete area-order term is the curvature of the relevant connection. This is the discrete algebraic origin of path dependence.

## 12. Geometric Computing Interpretation

The `coord` framework changes the primary representation of a geometric computation:

sampld position  $\rightarrow$  local measuring object  $\rightarrow$  relative object variation  $\rightarrow$  geometric invariants.

For computer-aided geometric design, the hierarchy becomes

point  $\rightarrow$  coord frame  $\rightarrow$  frame curve  $\rightarrow$  frame surface  $\rightarrow$  oriented boundary graph  $\rightarrow$  closed shell.

A contact sample may carry

$$F_i = (p_i, t_i, n_{A,i}, n_{B,i}, \Gamma_i, \varepsilon_i),$$

where  $p_i$  is position,  $t_i$  is the contact tangent,  $n_{A,i}, n_{B,i}$  are source-surface normals,  $\Gamma_i$  stores semantic provenance, and  $\varepsilon_i$  stores numerical residuals.

The boundary can then serve as the construction source of adjacent surface branches rather than as a later fitting target. Shared boundary samples are shared coordinate objects, so geometry, orientation, provenance, and residual information remain coupled throughout reconstruction.

This application illustrates the broader mathematical point: a `coord` is not merely a transformation container. It is a transportable local state of spatial measurement.

## 13. Scope and Mathematical Boundaries

### 13.1. coord is not a new division algebra

The quaternions form a normed division algebra. The `coord` object does not. It is a structured affine measuring-frame object derived from quaternionic rotation and additional spatial data. Its multiplication is transformation composition, and its division is relative transformation, not number-theoretic division.

### 13.2. Anisotropic coord objects are not generally group-closed

The diagonal-scale representation is exact for orthogonal measuring frames. Arbitrary composition may create shear. Exact closure requires uniform scale, a restricted commuting case, or extension to  $GL^+(3)$ .

### 13.3. Full ambient logarithmic variation is pure gauge

The full field

$$G_\mu = (\partial_\mu \hat{C}) \hat{C}^{-1}$$

satisfies its Maurer–Cartan identity. Nonzero surface curvature belongs to the induced tangent connection, not directly to the unprojected ambient logarithmic derivative.

### 13.4. Integration is ordered

Because coordinate transformations do not generally commute,

$$\exp\left(\int G\right)$$

cannot replace

$$\mathcal{D}\exp\left(\int G\right)$$

unless commutativity is proved. Path ordering is not a technical inconvenience; it is the algebraic carrier of spatial path dependence.

### 13.5. Global reconstruction requires compatibility

Local differential data do not automatically define a single global object. Maurer–Cartan compatibility, tangent-field closure, transition consistency, singularities, and topology determine whether globalization is path independent and globally possible.

### 13.6. Complexification is a separate extension

The real quaternionic derivation of `coord` is complete at the level developed here. Algebraic complexification gives

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C}),$$

but it does not by itself imply a  $U(3)$  field, a physical gauge theory, or a relativistic field equation. Such extensions require an explicit representation space, gauge action, reality condition, variational principle, and observable map.

## 14. Conclusion

The computable coordinate system is the natural scale-bearing affine completion of quaternionic orientation:

$$\mathbb{H} \supset S^3 = \text{Spin}(3) \longrightarrow SO(3) \longrightarrow \text{UCoord}^+(3) \longrightarrow C_{\perp}^+(3).$$

The successive additions have precise meanings:

|                           |                                  |
|---------------------------|----------------------------------|
| unit quaternion           | → orientation,                   |
| quaternion norm           | → uniform scale,                 |
| affine origin             | → location,                      |
| independent ruler lengths | → anisotropic local measurement, |
| field dependence          | → spatial calculus.              |

Once the coordinate system is promoted from passive background to first-class object, spatial calculus acquires a direct object language:

multiplication = finite spatial composition,  
 division = relative localization,  
 Log difference = spatial differentiation,  
 Exp/product integral = spatial integration and globalization,  
 closed-loop product = holonomy and curvature response.

The most concise statement of the theory is therefore:

*A computable coordinate system is a local spatial measuring object. Its logarithmic relative variation is the differential datum of space, and its ordered composition is the integral reconstruction of space.*

This viewpoint does not discard classical differential geometry. It reorganizes it around a computational object that already carries the position, orientation, scale, and local-to-global conversion required by geometric algorithms. Curvature is one consequence of this object calculus; curve and surface reconstruction, transport, closure auditing, and geometric topology are others.

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